

RIGOROUS DYNAMICS AND RADIATION THEORY FOR A PAULI-FIERZ MODEL IN THE ULTRAVIOLET LIMIT

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ABSTRACT. The present paper is devoted to the detailed study of quantization and evolution of the point limit of the Pauli-Fierz model for a charged oscillator interacting with the electromagnetic field in dipole approximation. In particular, a well defined dynamics is constructed for the classical model, which is subsequently quantized according to the Segal scheme. To this end, the classical model in the point limit, already obtained in [20], is reformulated as a second order abstract wave equation, and a consistent quantum evolution is given. This allows a study of the behaviour of the survival and transition amplitudes for the process of decay of the excited states of the charged particle, and the emission of photons in the decay process. In particular, for the survival amplitude the exact time behaviour is found. This is completely determined by the resonances of the systems plus a tail term prevailing in the asymptotic, long time regime. Moreover, the survival amplitude exhibits in a fairly clear way the Lamb shift correction to the unperturbed frequencies of the oscillator.

1. INTRODUCTION

In recent years a considerable effort was tributed by the mathematical physics community to the problem of a rigorous formulation of

the dynamics of the main models in nonrelativistic quantum field theory. In particular, a comprehensive study of the Pauli-Fierz model, the model which describes the low energy interaction of nonrelativistic matter and electromagnetic radiation, was undertaken by various authors, both in its full form, or making resort to different approximations, such as rotating wave approximation, dipole approximation, or others. Correspondingly, a wealth of results concerning various aspects of the model have been obtained, concerning self-adjointness of the hamiltonian, existence, multiplicity or also non existence of the ground state and related infrared behaviour, and detailed study of the spectral properties of the model and of its resonances. In the present paper we give a comprehensive analysis of some of the previous problems in the special case of a *point* charged oscillator interacting with the electromagnetic field in dipole approximation. While this model is unrealistically simple compared to the case, to give an example, of the hydrogen atom interacting with the full (not dipole) radiation field, (about which a lot is known thanks to the work of Lieb et al. [12],[16] and Fröhlich et al. [3],[4], the novelty of the present work resides in the fact that we are able to cope with the point limit of the model. The removal of the ultraviolet cutoff in the interaction between matter and radiation is in its generality, a difficult and unsolved problem, and in particular one not faced off in the quoted rigorous literature. In some previous papers of the last two authors ([19, 20] and references therein) the *renormalized* dynamics of the classical Pauli-Fierz model in dipole

approximation ([21]) and for fairly general external potentials, was rigorously constructed and analyzed. In particular, it was shown that the evolution of the Pauli-Fierz model in the point limit is given by an abstract wave equation generated by a family of operators related to the so called point interactions (see [1] and references therein). Given the classical model in the ultraviolet limit, a second step would be to construct the quantized model. The harmonic potential has the unique feature of giving rise to linear equations of motion, so that the classical equations for the system take the form of an abstract *linear* wave equation. This allows a plain quantization *à la* Segal of the model, which seems otherwise quite problematic (in contrast with the case of the regularized Pauli-Fierz model, where canonical quantization works: a detailed study of the regularized dipole Pauli-Fierz model with an external harmonic potential was given by Arai in [2]). So we confine ourselves to this case, about which we give in the next section a self-contained treatment independent and different in spirit from [20], to clarify some of the themes discussed above. We outline briefly our main results. The classical evolution of the system is given by the abstract wave equation

$$(\partial_{tt}^2 + L_e)\xi = 0$$

for the couple $\xi = (A, p)$ where A is the vector potential of the electromagnetic field in Coulomb gauge and p is the particle momentum variable (see section 2 for the explanation of this choice). Here L_e is a self-adjoint operator in $L_*^2(\mathbb{R}^3) \oplus \mathbb{R}^3$ ($*$ stands for “divergenceless”)

such that its resolvent can be explicitly calculated (see Lemma 2.2 and Theorem 2.3).

The operator $L_{\mathbf{e}}$ has a single negative eigenvalue and the rest of its spectrum is purely absolutely continuous and coincides with $[0, +\infty)$. Thus, to quantize the abstract wave equation above according to the Segal method, one has to take $L_{\mathbf{e}}^+$, i.e. $L_{\mathbf{e}}$ projected onto the spectral subspace corresponding to $\sigma_{ac}(L_{\mathbf{e}})$. The (first) quantized dynamics of the system is defined through the Schrödinger-like equation $i\dot{\psi} = (L_{\mathbf{e}}^+)^{1/2}\psi$, defined on the complex Hilbert space $L_*^2(\mathbb{R}^3; \mathbb{C}^3) \oplus \mathbb{C}^3$ and the hamiltonian of the quantized system of particle and field is given by the second quantization $d\Gamma((L_{\mathbf{e}}^+)^{\frac{1}{2}})$, on the Fock space over $L_*^2(\mathbb{R}^3; \mathbb{C}^3) \oplus \mathbb{C}^3$.

A preliminary but essential step in the description of the properties of the system is to write the first quantized evolution in terms of the resolvent of the original *classical* operator $L_{\mathbf{e}}^+$. By Stone formula, spectral theorem and after some work, one gets (see Lemma 2.5) a representation (perhaps new or at least not known to us) for the transition amplitudes $\langle \psi_1, e^{-it(L_{\mathbf{e}}^+)^{1/2}} \psi_2 \rangle$ between one particle states in terms of the resolvent of $L_{\mathbf{e}}$. Correspondingly, one has an expression for the survival and transition amplitudes for the second quantized model on the Fock space, just using functoriality of Γ . Our main concern here is in the calculation of two relevant characteristics of the evolution. The first is the survival amplitude $\mathcal{S}(t)$ of the unperturbed first excited bound state of the oscillator. The second is the amplitude transition $\mathcal{A}(t)$ between states with the photon field in the vacuum state and the oscillator in the first excited state, and states with one photon and the

oscillator in the ground state. The survival and transition amplitudes relative to more general states factorize in a sum of product of these two simpler types. The survival amplitude $\mathcal{S}(t)$ has a particularly neat form (see Theorem 3.3):

$$\mathcal{S}(t) = c_1 e^{-\lambda_e |t|} + c_2 e^{-\gamma_e |t|} e^{-i\omega_e t} + c_3 e^{-\lambda_e |t|} \text{Ei}(\lambda_e |t|) .$$

In this formula, the complex numbers

$$z_k := (-1)^k \omega_e - i\gamma_e, \quad k = 1, 2$$

coincide with the complex poles of the analytically continued resolvent of L_e^+ ; $-\lambda_e^2$ is the unique negative eigenvalue of L_e , and by Ei we mean the exponential integral function. The complex numbers c_1 , c_2 and c_3 depend on the physical parameters. So, in the time evolution of the survival amplitude for the bound states of the oscillator it is possible to distinguish three different time behaviours. The first term, depends on a resonance on the imaginary axis originated from the projection onto the positive spectral subspace; it is a pure exponentially decaying term (for positive times) and the characteristic time of the decay, for realistic values of mass and charge of the electron, has an order of 10^{-23} sec, an exceedingly small time. The second term is an exponentially damped oscillation described in terms of the complex resonance poles. In particular, the imaginary part of the resonance γ_e gives as usual a measure of the lifetime of the excited unperturbed states, or, equivalently, the breadth of the spectral lines of the spontaneous decay of the excited states (according to the Breit Wigner law, see [8]); while

the real part gives the position of the maximum in the emission of the spectral line. In terms of given physical parameters of the systems, the behaviour of these quantities is the following:

$$\omega_e = \omega_0 + \frac{28\omega_0^3}{3m^2c^6} e^4 + O(e^5), \quad \gamma_e = \frac{2\omega_0^2}{3mc^3} e^2 + O(e^5)$$

Here the symbols m , c and e denote the (renormalized) phenomenological mass, the velocity of light and the electric charge respectively. The noteworthy fact is the Lamb shift in the expression of ω_e . The maximum in the emission does not appear in correspondence of the unperturbed frequency of the oscillator, but at a displaced frequency. The last term, taking into account the asymptotic behaviour of the exponential integral, is of the order $\frac{1}{t}$ for $t \gg 1$. The appearance of a slowly decaying tail implies a departure from the purely exponential decay given by the Breit Wigner law, and is well known both in theoretical models and experimental studies. See Lemma 2.5 and the following remarks for an interpretation of its origin.

The transition amplitudes $\mathcal{A}(t)$ have a form very similar to the one for $\mathcal{S}(t)$ (see Theorem 3.5), and to them apply the same remarks and comments concerning their time behaviour.

2. AN EXACTLY SOLUBLE MODEL IN CLASSICAL AND QUANTUM ELECTRODYNAMICS

2.1. Classical theory. The classical Pauli-Fierz model for a particle with charge e , charge density $e\rho_r$ and bare mass m_r interacting with the electromagnetic field in dipole approximation and subjected to a

restoring harmonic force, is described by the hamiltonian

$$H = 2\pi c^2 \langle E, E \rangle + \frac{1}{8\pi} \langle A, \Delta A \rangle + \frac{1}{2m_r} \left| p - \frac{\mathbf{e}}{c} \langle \rho_r, A \rangle \right|^2 + \frac{1}{2} \alpha |q|^2 ,$$

where $E = \dot{A}/(4\pi c^2)$ is the canonical variable conjugated to A and p is the canonical momentum conjugated to the particle position q . The model is written in the Coulomb gauge, so the field are divergenceless. A suitable phase space for this dynamical system is $H_*^1(\mathbb{R}^3) \oplus L_*^2(\mathbb{R}^3) \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ where $H_*^1(\mathbb{R}^3)$ denotes the space of divergenceless locally square integrable vector fields with square integrable first derivatives. A point electron should have a Dirac measure as charge distribution, and the regular form factor ρ_r is introduced to give meaning to the equations of motion. So r has to be interpreted as a measure of the particle radius. In the point limit, as $r \downarrow 0$, the charge distribution ρ_r weakly converges to δ_0 , and the Hamilton equations corresponding to the Pauli-Fierz hamiltonian loose their original meaning. A well defined dynamical system is recovered only at the exence of renormalizing the bare mass m_r . This procedure is analyzed in detail in [19] and we content ourself to say here that the correct prescription is given by the well known relation between the bare, electromagnetic (m_{em}) and renormalized (m) masses,

$$m = m_r + m_{em} = m_r + \frac{8\pi}{3} \frac{\mathbf{e}^2}{c^2} \langle (-\Delta)^{-1} \rho_r, \rho_r \rangle .$$

Keeping m fixed to the physical value, this is the only choice for the bare mass which allows to obtain a nontrivial limit for the Pauli-Fierz model.

From now on, with the symbol m_r we mean precisely the function $m_r(\mathbf{e}, m)$ given by the above relation.

The rigorous deduction of the limit dynamics is carried out in [20] for a general external potential; specializing the result to the case of the harmonic potential one obtains the equations

$$\begin{aligned}\dot{A} &= 4\pi c^2 E \\ \dot{E} &= -\frac{1}{4\pi} H_m^p A \\ \dot{q} &= Q_A \\ \dot{p} &= -\alpha q .\end{aligned}$$

The p -dependent operator H_m^p is an affine deformation of a linear self-adjoint operator in the class of point interactions (see [1] and references therein for the use of point interactions in quantum mechanics): H_m^p is linear if and only if $p = 0$ and H_m^0 is the vector-valued version of one of such linear operators; Q_A is a certain linear functional which in some sense extract the singular part of the vector potential.

While obtaining a well defined dynamics in the point limit it is an interesting and not obvious result, it remains unclear how to quantize such a system. So, according to the point of view we adopt in this paper, and in view of the Segal quantization of the system, we would like to work with (abstract) second order wave equations of the form

$$\ddot{\xi}(t) = -L\xi(t) ,$$

with a suitable self-adjoint operator L . This is not the case of the previous system of equation, due to the presence of a tight relation between the dynamical variables contained in the definition of the operator H_m^p . Going back to the regularized system, the corresponding second order equations are not better from this point of view; they are

$$\begin{aligned}\frac{1}{c^2}\ddot{A} &= \Delta A + \frac{4\pi\mathbf{e}}{c} M\dot{q}\rho , \\ m_r\ddot{q} &= -\frac{\mathbf{e}}{c} \langle \rho_r, \dot{A} \rangle - \alpha q ,\end{aligned}$$

and the presence of \dot{q} and \dot{A} on the right hand side make them not of the desired form. So we prefer to give an construction of the limit dynamics independent of the one given in [20]. In fact, making resort to the hamiltonian regularized equations above, it is simple to overcome this problem. Deriving with respect to time the last equation and using the third, one obtains that the couple $\xi = (A, p)$ satisfies the abstract second order wave equation

$$\ddot{\xi} = -L_{\mathbf{e}}^r \xi$$

where the operator $L_{\mathbf{e}}^r$ is given by

$$L_{\mathbf{e}}^r(A, p) = \left(-c^2 \Delta A - \frac{4\pi\mathbf{e}c}{m_r} M \left(p - \frac{\mathbf{e}}{c} \langle \rho_r, A \rangle \right) \rho_r, \frac{\alpha}{m_r} \left(p - \frac{\mathbf{e}}{c} \langle \rho_r, A \rangle \right) \right)$$

The analogous result is obtained by means of the canonical transformation which exchanges the particle position and momentum (see also [26])

$$q = -P , \quad p = Q .$$

The change of dynamical variable from q to p in the second order (or lagrangian) formalism is not particularly relevant to the analysis of the problems we are interested in, and the interpretation of the results we get. For example, the time behaviour of the classical position q , which is important in the calculation of the survival amplitude of the bound states of the oscillator, is quite simply related to the time behaviour of the variable p : up to a constant, the derivative of the momentum gives the position, and this relation is preserved in the limit dynamics.

Another important point to note, is that the operator $L_{\mathbf{e}}^r$ is a finite rank perturbation of the noninteracting operator $L_0 = (-c^2\Delta, 0)$. This simple structure suggests the possibility that the operator $L_{\mathbf{e}}^r$ has a limit for $r \downarrow 0$, and being an unbounded operator such a limit, if existing, should be sought in the resolvent sense. The calculation of the resolvent of the operator $L_{\mathbf{e}}^r$ is lengthy but elementary. We omit the proof and give the result in the following

Lemma 2.1. *For every $z \in \mathbb{C}_{\pm}$, the resolvent of $L_{\mathbf{e}}^r$ is given by*

$$(L_{\mathbf{e}}^r - z^2)^{-1}(A, p) = (\mathcal{G}_z^{\pm} * A, 0) - \Lambda^r(z) R_{\mathbf{e}}^r(z)(A, p),$$

where

$$\Lambda^r(z) = \frac{-1}{m_r k_1^r k_2^r}, \quad \mathcal{G}_z^{\pm}(x) = \frac{1}{c^2} \frac{e^{\pm iz|x|/c}}{4\pi|x|}, \quad \pm \text{Im}z > 0,$$

$$R_{\mathbf{e}}^r(z)(A, p) = (R_{\mathbf{e}}^{1r}(z)(A, p), R_{\mathbf{e}}^{2r}(z)(A, p)) ,$$

$$\begin{aligned}
R_{\mathbf{e}}^{1r}(z)(A, p) &= \frac{4\pi\mathbf{e}}{c} M \left((z^2 c \mathbf{e} \langle (-c^2\Delta - z^2)^{-1} A, \rho_r \rangle \right. \\
&\quad \left. + c^2 \frac{m_r k_1^r}{\alpha} (z^2 + k_2^r) p \right) (-c^2\Delta - z^2)^{-1} \rho_r \\
R_{\mathbf{e}}^{2r}(z)(A, p) &= \alpha \frac{\mathbf{e}}{c} \langle (-c^2\Delta - z^2)^{-1} A, \rho_r \rangle + m_r k_1^r p,
\end{aligned}$$

$$\begin{aligned}
k_1^r &= 1 + \frac{8}{3} \pi \mathbf{e}^2 \langle (-c^2\Delta - z^2)^{-1} \rho_r, \rho_r \rangle, \\
k_2^r &= \frac{\alpha}{m_r} - z^2 - \frac{8}{3} \frac{\alpha \pi \mathbf{e}^2}{m_r^2 k_1^r} \langle (-c^2\Delta - z^2)^{-1} \rho_r, \rho_r \rangle.
\end{aligned}$$

The next step is the point limit $r \downarrow 0$. The result is analogous to the corresponding result given in [19] for the case of a free particle, and the proof is modelled on one of the well known ways of defining point interactions (see [1]). We give only an outline of the proof.

Lemma 2.2. *Let $\rho_r \rightarrow \delta_0$ weakly as $r \downarrow 0$. For every fixed $z \in \mathbb{C}_{\pm}$, $(L_{\mathbf{e}}^r - z^2)^{-1}$ converges as $r \downarrow 0$ in the norm resolvent sense to the operator*

$$(\mathcal{G}_z^{\pm} * A, 0) - \Lambda_{\pm}(z) R_{\mathbf{e}}^{\pm}(z)(A, p),$$

where, putting $\omega_0^2 := \frac{\alpha}{m}$,

$$\begin{aligned}
R_{\mathbf{e}}^{\pm}(z)(A, p) &= \left(\frac{4\pi\mathbf{e}}{c} M (z^2 c \mathbf{e} \langle \mathcal{G}_{z^*}^{\mp}, A \rangle + c^2 p) \mathcal{G}_z^{\pm}, \right. \\
&\quad \left. m\omega_0^2 \frac{\mathbf{e}}{c} \langle \mathcal{G}_{z^*}^{\mp}, A \rangle + \left(m \pm i \frac{2\mathbf{e}^2}{3c^3} z \right) p \right),
\end{aligned}$$

$$\Lambda_{\pm}(z) = \frac{1}{\pm i \frac{2\mathbf{e}^2}{3c^3} z^3 - m(\omega_0^2 - z^2)}.$$

Such an operator is the resolvent of a self-adjoint operator $L_{\mathbf{e}}$ on $L_*^2(\mathbb{R}^3) \oplus \mathbb{R}^3$ when on the component \mathbb{R}^3 one considers the scalar product $\langle p_1, p_2 \rangle := \kappa_0 p_1 \cdot p_2$, with $\kappa_0 := \frac{4\pi c^2}{m\omega_0^2}$.

Proof. The proof of the convergence of the regularized resolvent is a direct consequence of the following limiting relations:

$$\begin{aligned} \lim_{r \downarrow 0} (-c^2 \Delta - z^2)^{-1} \rho_r &= \mathcal{G}_z^\pm, \\ \lim_{r \downarrow 0} k_1^r &= 0, \quad \lim_{r \downarrow 0} k_1^r m_r = m \pm i \frac{2e^2}{3c^3} z, \\ \lim_{r \downarrow 0} k_2^r &= -\frac{\pm i \frac{2e^2}{3c^3} z^3 - m(\omega_0^2 - z^2)}{m \pm i \frac{2e^2}{3c^3} z}. \end{aligned}$$

Checking that the limit operator is the resolvent of a self-adjoint operator is routine. See [19] for a similar verification. \square

By the resolvent just constructed it is straightforward to derive the actions of L_e itself and of its spectral properties:

Theorem 2.3. *The action and domain of the self-adjoint operator*

$$L_e : D(L_e) \subseteq L_*^2(\mathbb{R}^3) \oplus \mathbb{R}^3 \rightarrow L_*^2(\mathbb{R}^3) \oplus \mathbb{R}^3$$

are given by

$$\begin{aligned} L_e(A, p) &= \left(-c^2 \Delta A_0, \omega_0^2 \left(p - \frac{e}{c} A_0(0) \right) \right), \\ D(L_e) &= \left\{ (A, p) \in L_*^2(\mathbb{R}^3) \oplus \mathbb{R}^3 : A = A_0 + \frac{4\pi e}{c} Mv \mathcal{G}, \right. \\ &\quad \left. \sqrt{-\Delta} A_0, \Delta A_0 \in L_*^2(\mathbb{R}^3), v \in \mathbb{R}^3, mv = p - \frac{e}{c} A_0(0) \right\}, \end{aligned}$$

where

$$\mathcal{G}(x) = \frac{1}{4\pi|x|}.$$

Moreover,

$$\sigma_p(L_e) = \{-\lambda_e^2\}, \quad \sigma_{ess}(L_e) = \sigma_{ac}(L_e) = [0, +\infty), \quad \sigma_{sc}(L_e) = \emptyset,$$

where

$$\lambda_e = \frac{3mc^3}{2e^2} + O(e^2)$$

is the unique real (and positive) solution of the third order equation

$$\frac{2e^2}{3c^3} \lambda^3 - m(\omega_0^2 + \lambda^2) = 0.$$

We emphasize that, due to the fact that the operator L_e has been constructed as a norm resolvent limit of the operator L_e^r , the flow generated by the limit operator coincides with the limit of the regularized flow, in the relevant norms of the phase space and uniformly in time. This allows to consider L_e as the generator of the limit dynamics. As a second remark, note that the algebraic equation $\Lambda_{\pm}(z)^{-1} = 0$ is nothing but the characteristic equation of the Abraham-Lorentz equation (see e.g. [17]) in the particular case of an harmonic external force, i.e.

$$-\tau_0 \ddot{q} + \ddot{q} + \omega_0 q = 0 \quad \tau_0 = \frac{2e^2}{3mc^3},$$

the equation which classically describes the behaviour of the particle position in the nonrelativistic regime (the classical relativistic equation was obtained by P.A.M.Dirac in [10]). In particular, the negative eigenvalue of L_e corresponds to the so called runaway solution of the A-L equation. In the traditional approaches, this solution is discarded due to its unphysical character. From the present point of view, we give up the (important) interpretative problem related to the presence of these instabilities, and take the attitude according to which the suppression of runaway behaviour corresponds to reduction of the dynamics on the

stable subspace, or equivalently, restriction to the absolutely continuous component of the spectrum. This should correspond, in ordinary scattering theory for Schrödinger operators, to the elimination of bound states. The procedure to obtain this reduction can be explicitly performed as follows. Let us consider, from now on, L_e as acting on the complex Hilbert space $L_*^2(\mathbb{R}^3; \mathbb{C}^3) \oplus \mathbb{C}^3$. It is self-adjoint when on the component \mathbb{C}^3 one considers the scalar product $\langle \zeta_1, \zeta_2 \rangle := \kappa_0 \zeta_1^* \cdot \zeta_2$.

Given $\zeta_1, \zeta_2, \zeta_3$, an orthonormal base in \mathbb{C}^3 , and defining

$$\mathcal{G}_{\lambda_e}(x) := \frac{1}{c^2} \frac{e^{-\lambda_e |x|/c}}{4\pi|x|}, \quad \kappa := \left(\frac{2\pi e^2 \lambda_0^3}{m^2 \omega_0^4 c} + \kappa_0 \right)^{1/2}, \quad \kappa_1 := \frac{e}{c} \lambda_e^2 \kappa_0,$$

let

$$\psi_i^0 = \frac{1}{\kappa} (-\kappa_1 M \zeta_i \mathcal{G}_{\lambda_e}, \zeta_i), \quad i = 1, 2, 3$$

be the normalized eigenvectors corresponding to the eigenvalue $-\lambda_e^2$. Then the projection P_e onto the absolutely continuous subspace of the operator L_e is given by

$$P_e \psi \equiv \psi^+ = \psi - \sum_{i=1,2,3} \langle \psi, \psi_i^0 \rangle \psi_i^0.$$

In particular

$$P_e(0, \zeta) = \frac{\kappa_2}{\kappa^2} \left(\frac{12\pi c^4}{e} M \zeta \mathcal{G}_{\lambda_e}, \lambda_e \zeta \right), \quad \kappa_2 := \frac{4\pi e^2 \lambda_e^2}{m^2 \omega_0^4 c}.$$

We define the positive self-adjoint operator L_e^+ by $L_e^+ := P_e L_e$, and from now on we consider this reduced operator as the generator of the physical limit dynamics.

2.2. Quantum theory. The operator L_e^+ generates the classical evolution, whereas, according to the results summarized in the appendix, the corresponding quantum evolution is given in terms of its square root $(L_e^+)^{1/2}$. More explicitly, denoting by $\mathcal{F}(L_*^2(\mathbb{R}^3; \mathbb{C}^3) \oplus \mathbb{C}^3)$ the bosonic Fock space over $L_*^2(\mathbb{R}^3; \mathbb{C}^3) \oplus \mathbb{C}^3$, i.e.

$$\mathcal{F}(L_*^2(\mathbb{R}^3; \mathbb{C}^3) \oplus \mathbb{C}^3) := \bigoplus_{n \geq 0} S_n(L_*^2(\mathbb{R}^3; \mathbb{C}^3) \oplus \mathbb{C}^3)^{\otimes n},$$

where S_n denotes the symmetrization operator on the n -th sector, the quantum hamiltonian on $\mathcal{F}(L_*^2(\mathbb{R}^3; \mathbb{C}^3) \oplus \mathbb{C}^3)$ corresponding to the second quantization of the classical wave equation

$$(\partial_{tt}^2 + L_e^+)(A, p) = 0$$

is given by

$$H_e := \hbar d\Gamma((L_e^+)^{1/2}).$$

Note that the noninteracting hamiltonian

$$H_0 := \hbar d\Gamma(L_0^{1/2}) \equiv \hbar d\Gamma(\sqrt{-\Delta} \oplus \omega_0)$$

is unitarily equivalent to

$$\hbar d\Gamma(\sqrt{-\Delta}) \otimes 1 + 1 \otimes \hbar d\Gamma(\omega_0),$$

defined on the Hilbert space

$$\mathcal{F}(L_*^2(\mathbb{R}^3; \mathbb{C}^3)) \otimes \mathcal{F}(\mathbb{C}^3) := \bigoplus_{n \geq 0} S_n L_*^2(\mathbb{R}^3; \mathbb{C}^3)^{\otimes n} \otimes \bigoplus_{n \geq 0} S_n(\mathbb{C}^3)^{\otimes n}.$$

The unitary operator

$$U : \mathcal{F}(L_*^2(\mathbb{R}^3; \mathbb{C}^3) \oplus \mathbb{C}^3) \rightarrow \mathcal{F}(L_*^2(\mathbb{R}^3; \mathbb{C}^3)) \otimes \mathcal{F}(\mathbb{C}^3) \quad (1)$$

giving the stated equivalence is defined by

$$\mathbf{U}\Omega := \Omega \otimes \Omega, \quad \mathbf{U}\mathbf{C}(\psi)\mathbf{U}^{-1} = \mathbf{C}(\varphi) \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{C}(\zeta),$$

where Ω denotes the vacuum, $\psi = (\varphi, \zeta)$ and \mathbf{C} is the usual creation operator. Moreover $\hbar d\Gamma(\omega_0)$ is unitarily equivalent to the usual harmonic oscillator hamiltonian on $L^2(\mathbb{R}^3; \mathbb{C})$ given by the self-adjoint operator

$$-\frac{\hbar^2}{2m} \Delta + \frac{m\omega_0^2}{2} q^2.$$

Concerning the interacting Hamiltonian \mathbf{H}_e , the present one is its first *explicit* construction in the ultraviolet limit and the problem arises if other representations more directly confrontable with usual canonical formulation could be given. We emphasize the fact that in our description, already at the classical level, the point limit produces an intimate interlacing between field singularities and particle variables, through the definition of the domain of the operator L_e itself, and this fact introduces essential difficulties in tracing the relation with the canonical formalism based on the usual regularized Pauli-Fierz Hamiltonian, where this constraint disappears. Nevertheless, in the quoted Arai paper ([2]), a reconstruction theorem based on the limit of the Wightman functions of the regularized model is outlined. It could be interesting to indagate the relations between two approaches.

Since the resolvent of L_e is quite explicit, making use of the Birman-Kato invariance principle to deal with the group generated by the square root, and of the Birman-Kuroda completeness theorem (see [7])

which is applicable because $(L_{\mathbf{e}} + z)^{-1} - (L_0 + z)^{-1}$ is a finite rank operator, one immediately obtains the following

Theorem 2.4. *Let P_0 and $P_{\mathbf{e}}$ be the orthogonal projections onto the absolutely continuous subspaces of L_0 and $L_{\mathbf{e}}$ respectively. Then the Möller wave operators exist, they are complete and,*

$$\Omega_{\pm}(\mathbf{H}_{\mathbf{e}}, \mathbf{H}_0) := \text{s-}\lim_{t \rightarrow \pm\infty} e^{-it\mathbf{H}_{\mathbf{e}}/\hbar} e^{-it\mathbf{H}_0/\hbar} \Gamma(P_0) = \Gamma(\Omega_{\pm}(L_{\mathbf{e}}, L_0))$$

$$\Omega_{\pm}(\mathbf{H}_0, \mathbf{H}_{\mathbf{e}}) := \text{s-}\lim_{t \rightarrow \pm\infty} e^{-it\mathbf{H}_0/\hbar} e^{-it\mathbf{H}_{\mathbf{e}}/\hbar} \Gamma(P_{\mathbf{e}}) = \Gamma(\Omega_{\pm}(L_0, L_{\mathbf{e}}))$$

A final result of this paragraph is a formula for the evaluation of transition amplitudes of the Schrödinger-like propagator $e^{-it(L_{\mathbf{e}}^+)^{1/2}}$, that is the scalar product of the type $\langle \psi_1, e^{-it(L_{\mathbf{e}}^+)^{1/2}} \psi_2 \rangle$, in terms of boundary values of the resolvent of the classical operator $L_{\mathbf{e}}$. In the formula the special form of the operator $L_{\mathbf{e}}$ plays no rule, and it holds true for a positive generator A whatsoever. By linearity it will be sufficient to suppose that ψ_1 and ψ_2 are real valued. Since (see the appendix for the definition of $W_{L_{\mathbf{e}}^+}$)

$$((iW_{L_{\mathbf{e}}^+} - z)^{-1})(\psi, 0) = (z(L_{\mathbf{e}}^+ - z^2)^{-1}\psi, -i\psi - iz^2(L_{\mathbf{e}}^+ - z^2)^{-1}\psi),$$

by Lemma A.2 and Lemma A.3 one obtains

$$\begin{aligned}
& \langle \psi_1, e^{-it(L_{\mathbf{e}}^+)^{1/2}} \psi_2 \rangle \\
&= \lim_{a \uparrow \infty} \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{-a}^a d\lambda \, e^{-it\lambda} (\langle \psi_1, ((\lambda + i\epsilon)(L_{\mathbf{e}} - (\lambda + i\epsilon)^2)^{-1} \\
&\quad - (\lambda - i\epsilon)(L_{\mathbf{e}} - (\lambda - i\epsilon)^2)^{-1}) \psi_2^+ \rangle) \\
&+ \lim_{a \uparrow \infty} \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{-a}^a d\lambda \, e^{-it\lambda} (\langle \psi_1, ((\lambda + i\epsilon)^2(L_{\mathbf{e}} - (\lambda + i\epsilon)^2)^{-1} \\
&\quad - (\lambda - i\epsilon)^2(L_{\mathbf{e}} - (\lambda - i\epsilon)^2)^{-1})(L_{\mathbf{e}}^+)^{-1/2} \psi_2^+ \rangle).
\end{aligned}$$

Moreover, using first resolvent identity and

$$(L_{\mathbf{e}}^+)^{-1/2} = \frac{1}{\pi} \int_{\mathbb{R}} ds (L_{\mathbf{e}}^+ + s^2)^{-1}$$

it turns that

$$\begin{aligned}
& \langle \psi_1, (L_{\mathbf{e}} - z^2)^{-1} (L_{\mathbf{e}}^+)^{-1/2} \psi_2^+ \rangle \\
&= \frac{1}{\pi} \int_{\mathbb{R}} \frac{ds}{s^2 + z^2} \langle \psi_1, (L_{\mathbf{e}} - z^2)^{-1} \psi_2^+ \rangle - \frac{1}{\pi} \int_{\mathbb{R}} \frac{ds}{s^2 + z^2} \langle \psi_1, (L_{\mathbf{e}} + s^2)^{-1} \psi_2^+ \rangle
\end{aligned}$$

Thus for any couple ψ_1, ψ_2 for which the limits

$$\langle \psi_1, (L_{\mathbf{e}} - \lambda_{\pm}^2)^{-1} \psi_2^+ \rangle := \lim_{\epsilon \downarrow 0} \langle \psi_1, (L_{\mathbf{e}} - (\lambda \pm i\epsilon)^2)^{-1} \psi_2^+ \rangle$$

exist, one obtains the following

Lemma 2.5.

$$\begin{aligned}
& \langle \psi_1, e^{-it(L_e^+)^{1/2}} \psi_2 \rangle \\
&= \lim_{a \uparrow \infty} \frac{1}{2\pi i} \int_{-a}^a d\lambda \, e^{-it\lambda} \lambda \left(\langle \psi_1, ((L_e - \lambda_+^2)^{-1} - (L_e - \lambda_-^2)^{-1}) \psi_2^+ \rangle \right) \\
&+ \frac{1}{\pi} \int_{\mathbb{R}} \frac{ds}{2\pi i} \int_{\mathbb{R}} d\lambda \, \frac{e^{-it\lambda} \lambda^2}{s^2 + \lambda^2} \langle \psi_1, ((L_e - \lambda_+^2)^{-1} - (L_e - \lambda_-^2)^{-1}) \psi_2^+ \rangle.
\end{aligned}$$

Note that the second contribution in the previous formula comes from the nonlocal relation between the real phase space classical variables and the complexified ones, described in Lemma A.3. Performing however the s integral one obtains, under the same condition of the previous result, the following alternative representation

$$\begin{aligned}
& \langle \psi_1, e^{-it(L_e^+)^{1/2}} \psi_2 \rangle \\
&= \lim_{a \uparrow \infty} \frac{1}{2\pi i} \int_0^a d\lambda \, e^{-it\lambda} \lambda \left(\langle \psi_1, ((L_e - \lambda_+^2)^{-1} - (L_e - \lambda_-^2)^{-1}) \psi_2^+ \rangle \right)
\end{aligned}$$

This last representation has a more direct meaning, in that it is an integral extended over the spectrum of the operator $(L_e^+)^{1/2}$. Moreover, it presents the evolution generated by $(L_e^+)^{1/2}$ as a Fourier transform of a function (for every fixed couple of states ψ_1 and ψ_2) supported on a half line. As a consequence of the Paley-Wiener theorem, the evolution of the amplitude transition cannot have a leading large time contribution of exponential type, but a slower one should appear (see [13],[14] for an early application of this remark to the time decay of amplitude transition). The exact time behaviour cannot be precised without the knowledge of further details about the generator. To this

end it is of more practical use the formula given in Lemma 2.5, as we see in the following section.

3. RADIATION THEORY

3.1. Generalities. In this section we want to give some details of the quantum dynamics of the model we are studying. In particular, we want to estimate, under the dynamics generated by H_e , the survival amplitudes of the bound states of H_0 and the probability amplitudes of the transitions between two of such states with emission of photons. We begin with some preliminaries on the general structure of the amplitude transitions in our model.

In the space $\mathcal{F}(L_*^2(\mathbb{R}^3; \mathbb{C}^3)) \otimes \mathcal{F}(\mathbb{C}^3)$ a bound state (n -level) for the hamiltonian operator H_0 is represented by a vector of the kind

$$\Omega \otimes S_n(\zeta_1 \otimes \cdots \otimes \zeta_n).$$

According to (1) this state is represented in $\mathcal{F}(L_*^2(\mathbb{R}^3; \mathbb{C}^3) \oplus \mathbb{C}^3)$ by the vector

$$S_n((0, \zeta_1) \otimes \cdots \otimes (0, \zeta_n))$$

By (1) again, a more general state with m photons, of the kind

$$S_m(\varphi_1 \otimes \cdots \otimes \varphi_m) \otimes S_n(\zeta_1 \otimes \cdots \otimes \zeta_n),$$

is represented in $\mathcal{F}(L_*^2(\mathbb{R}^3; \mathbb{C}^3) \oplus \mathbb{C}^3)$ by the vector

$$S_{m+n}((\varphi_1, 0) \otimes \cdots \otimes (\varphi_m, 0) \otimes (0, \zeta_1) \otimes \cdots \otimes (0, \zeta_n)).$$

Note that, being $H_e = \hbar d\Gamma((L_e^+)^{1/2})$, any sector in $\mathcal{F}(L_*^2(\mathbb{R}^3; \mathbb{C}^3) \oplus \mathbb{C}^3)$ is preserved under e^{-itH_e} . Thus only the transitions from a n -level to a m -level with the emission of $n - m$ photons are allowed.

On the other hand, as we shall see, asymptotically the particle part of the wavefunction of the state relaxes to the ground state, and this yields to the asymptotic conservation of the photon number in the scattering process. This property was devised by A.Arai in [2] for the regularized Pauli-Fierz model with quadratic potential, as a consequence of the factorization properties of the scattering matrix, and we find it again in the point model.

Since

$$\begin{aligned} e^{-itH_e/\hbar} S_n(\psi_1 \otimes \cdots \otimes \psi_n) &= \Gamma(e^{-it(L_e^+)^{1/2}}) S_n(\psi_1 \otimes \cdots \otimes \psi_n) \\ &= \frac{1}{n!} \sum_{\sigma} e^{-it(L_e^+)^{1/2}} \psi_{\sigma_1} \otimes \cdots \otimes e^{-it(L_e^+)^{1/2}} \psi_{\sigma_n}, \end{aligned}$$

the survival amplitude of the bound state $\Omega \otimes S_n(\zeta_1 \otimes \cdots \otimes \zeta_n)$ is given by

$$\frac{1}{n!} \sum_{\sigma} \prod_{j=1}^n \langle (0, \zeta_{\sigma_j}), e^{-it(L_e^+)^{1/2}} (0, \zeta_j) \rangle$$

whereas the probability amplitude of the transition

$$\Omega \otimes S_n(\zeta_1 \otimes \cdots \otimes \zeta_n) \leftrightarrow S_m(\varphi_1 \otimes \cdots \otimes \varphi_m) \otimes S_{n-m}(\xi_{m+1} \otimes \cdots \otimes \xi_{n-m})$$

is given by

$$\frac{1}{n!} \sum_{\sigma} \prod_{\sigma_j \leq m} \langle (\varphi_{\sigma_j}, 0), e^{-it(L_e^+)^{1/2}} (0, \zeta_j) \rangle \prod_{\sigma_k > m} \langle (0, \xi_{\sigma_k}), e^{-it(L_e^+)^{1/2}} (0, \zeta_k) \rangle.$$

3.2. Survival amplitudes. After these general remarks, we evaluate the survival amplitudes for the bound states of the unperturbed dynamics along the perturbed evolution. To simplify the exposition we break the analysis in a number of lemmata.

Lemma 3.1. *For any $\zeta_1, \zeta_2 \in \mathbb{C}^3$ one has*

$$\langle (0, \zeta_1), e^{-it(L_e^+)^{1/2}}(0, \zeta_2) \rangle = \mathcal{S}(t) \zeta_1^* \cdot \zeta_2,$$

where

$$\mathcal{S}(t) = -\frac{2\kappa_0\kappa_2}{\kappa^2} I(t)$$

and, with $\pm t > 0$,

$$I(t) = \mp e^{\mp\lambda_e t} i\lambda_e \frac{p(-i\lambda_e)}{q(-i\lambda_e)} \pm e^{\mp\gamma_e t} \frac{e^{-i\omega_e t} 2z_{\pm}}{z_{\pm} + i\lambda_e} \frac{p(z_{\pm})}{q'(z_{\pm})} + J_1(t) + J_2(t),$$

$$J_1(t) = \frac{1}{\pi} \int_{\mathbb{R}} ds \frac{e^{\mp|s|t}|s|}{2(|s| + \lambda_e)} \frac{p(i|s|)}{q(i|s|)},$$

$$J_2(t) = \frac{1}{\pi} \int_{\mathbb{R}} ds \left(\frac{e^{\mp|s|t}|s|}{2(|s| - \lambda_e)} \frac{p(-i|s|)}{q(-i|s|)} - \frac{e^{\mp\lambda_e t} \lambda_e^2}{s^2 - \lambda_e^2} \frac{p(-i\lambda_e)}{q(-i\lambda_e)} \right),$$

$$p(z) = \frac{e^2 \lambda_e}{3c^3} z + i \frac{m}{\lambda_e} \left(\omega_0^2 + \frac{\lambda_e^2}{2} \right),$$

$$q(z) = \frac{2e^2}{3c^3} z^2 + i \frac{m\omega_0^2}{\lambda_e^2} (z + i\lambda_e),$$

where

$$z_{\pm} := \pm\omega_e - i\gamma_e,$$

$$\omega_e = \omega_0 + \frac{28\omega_0^3}{3m^2c^6} e^4 + O(e^5), \quad \gamma_e = \frac{2\omega_0^2}{3mc^3} e^2 + O(e^5)$$

are the two roots of $q(z)$.

Proof. Let $z \in \mathbb{C}_\pm$. Taking in account the algebraic equation satisfied by λ_e and the identity

$$\langle \mathcal{G}_{z^*}^\mp, \mathcal{G}_{\lambda_e} \rangle = \frac{1}{4\pi c^3} \frac{1}{\lambda_e \mp iz}$$

we have, for the projected resolvent,

$$\begin{aligned} & \langle (0, \zeta_1), (L_e - z^2)^{-1}(0, \zeta_2)^+ \rangle \\ &= -\kappa_0 \frac{\kappa_2}{\kappa^2} \Lambda_\pm(z) \left(m\omega_0^2 8\pi c^3 \langle \mathcal{G}_{z^*}^\mp, \mathcal{G}_{\lambda_e} \rangle + \left(m \pm i \frac{2e^2}{3c^3} z \right) \lambda_e \right) \zeta_1^* \cdot \zeta_2 \\ &= \frac{-2\kappa_0 \kappa_2}{\kappa^2 (\pm iz - \lambda_e)} \frac{\left(\pm i \frac{e^2 \lambda_e}{3c^3} z - \frac{m}{\lambda_e} \left(\omega_0^2 + \frac{\lambda_e^2}{2} \right) \right) (\pm iz + \lambda_e)}{\left(\frac{2e^2}{3c^3} z^2 + \frac{m\omega_0^2}{\lambda_e^2} (\pm iz - \lambda_e) \right) (\pm iz + \lambda_e)} \zeta_1^* \cdot \zeta_2. \end{aligned}$$

Thus using the representation formula given in lemma 2.5, we get

$$\langle (0, \zeta_1), e^{-it(L_e^+)^{1/2}}(0, \zeta_2) \rangle = -\frac{2\kappa_0 \kappa_2}{\kappa^2} \left(I_1(t) + \frac{1}{\pi} \int_{\mathbb{R}} ds I_2(t, s) \right) \zeta_1^* \cdot \zeta_2,$$

where

$$\begin{aligned} I_1(t) &:= \frac{1}{2\pi i} \lim_{a \uparrow \infty} \int_{-a}^a d\lambda e^{-it\lambda} \lambda (f(\lambda) - f(-\lambda)), \\ I_2(t, s) &:= \frac{1}{2\pi i} \int_{\mathbb{R}} d\lambda e^{-it\lambda} \lambda^2 \frac{f(\lambda) - f(-\lambda)}{s^2 + \lambda^2}, \\ f(\lambda) &:= \frac{1}{\lambda + i\lambda_e} \frac{\frac{e^2 \lambda_e}{3c^3} \lambda + i \frac{m}{\lambda_e} \left(\omega_0^2 + \frac{\lambda_e^2}{2} \right)}{\frac{2e^2}{3c^3} \lambda^2 + i \frac{m\omega_0^2}{\lambda_e^2} (\lambda + i\lambda_e)} \equiv \frac{1}{\lambda + i\lambda_e} \frac{p(\lambda)}{q(\lambda)}. \end{aligned}$$

Then, by residue theorem and Jordan's lemma, when $\pm t > 0$, one obtains

$$I_1(t) = \mp e^{\mp \lambda_e t} i \lambda_e \frac{p(-i\lambda_e)}{q(-i\lambda_e)} \pm e^{\mp \gamma_e t} \sum_{k=1,2} \frac{e^{\mp(-1)^k i \omega_e t} z_k}{z_k + i\lambda_e} \frac{p(z_k)}{q'(z_k)},$$

where $z_k := (-1)^k \omega_e - \gamma_e$, and

$$\begin{aligned} I_2(t, s) = & - \frac{e^{\mp \lambda_e t} \lambda_e^2}{s^2 - \lambda_e^2} \frac{p(-i\lambda_e)}{q(-i\lambda_e)} + e^{\mp t \gamma_e} \sum_{k=1,2} \frac{z_k}{s^2 + z_k^2} \frac{e^{\mp(-1)^k i \omega_e t} z_k}{z_k + i\lambda_e} \frac{p(z_k)}{q'(z_k)} \\ & + e^{\mp |s| t} \frac{|s|}{2} \left(\frac{1}{|s| \mp \lambda_e} \frac{p(\mp i|s|)}{q(\mp i|s|)} + \frac{1}{|s| \pm \lambda_e} \frac{p(\pm i|s|)}{q(\pm i|s|)} \right) \end{aligned}$$

Since

$$\frac{1}{\pi} \int_{\mathbb{R}} ds \frac{z_k}{s^2 + z_k^2} = (-1)^k,$$

one has

$$\begin{aligned} I_1(t) + \frac{1}{\pi} \int_{\mathbb{R}} ds I_2(t, s) \\ = \mp e^{\mp \lambda_e t} i \lambda_e \frac{p(-i\lambda_e)}{q(-i\lambda_e)} \pm e^{\mp t \gamma_e} \frac{e^{-i \omega_e t} 2 z_k}{z_k + i \lambda_e} \frac{p(z_k)}{q'(z_k)} \\ + \frac{1}{\pi} \int_{\mathbb{R}} ds \left(\frac{e^{\mp |s| t} |s|}{2(|s| - \lambda_e)} \frac{p(-i|s|)}{q(-i|s|)} - \frac{e^{\mp \lambda_e t} \lambda_e^2}{s^2 - \lambda_e^2} \frac{p(-i\lambda_e)}{q(-i\lambda_e)} \right) \\ + \frac{1}{\pi} \int_{\mathbb{R}} ds \frac{e^{\mp |s| t} |s|}{2(|s| + \lambda_e)} \frac{p(i|s|)}{q(i|s|)}, \end{aligned}$$

where $k = 2$ if $t > 0$ and $k = 1$ if $t < 0$. \square

We distinguish three contributions in the previous formula for $I(t)$. The one on the first line is due to the resonances of the system. In particular, we have a purely exponential term with lifetime λ_e , coming from the projection on the stable subspace of the system, which dies out very quickly; and terms exponentially damped, due to the complex resonances. These last resonances are typical of the Breit-Wigner distribution, and they are the only terms which survive to the usual single pole approximation for the resolvent. They give both the breadth of emission lines in the spectrum (or equivalently the lifetime

of the process) and the Lamb shift, as recalled in the introduction. The other contributions to the survival amplitude need more direct analysis. To simplify the exposition we confine to the case of $t > 0$, the case $t < 0$ being completely analogous. The term $J_1(t)$ is nothing else than the Laplace transform (which we denote by the symbol \mathcal{L}) a rational function. It is a tedious but standard calculation to verify that such a Laplace transform is in fact the sum of a pure exponential and a damped oscillation with the same characteristic exponents as the ones coming from the first group of terms. So, the term $J_1(t)$ corresponds to a further resonance contribution.

Concerning $J_2(t)$ the following result holds true.

Lemma 3.2. *There exist complex constants c_1, c_2, c_3 such that, for $t > 0$*

$$J_2(t) = \frac{1}{\pi} \mathcal{L} \left(\frac{c_1 s + c_2}{(s - \gamma_e)^2 + \omega_e^2} \right) + \frac{c_3}{\pi} P.V. \int_0^\infty \frac{e^{-st}}{s - \lambda_e} ds$$

Proof. Introducing a parameter ϵ to isolate the singularity in λ_e , one can write

$$\begin{aligned} J_2(t) &= \frac{2}{\pi} \int_0^{+\infty} ds \left(\frac{e^{-|s|t}|s|}{2(|s| - \lambda_e)} \frac{p(-i|s|)}{q(-i|s|)} - \frac{e^{-\lambda_e t} \lambda_e^2}{s^2 - \lambda_e^2} \frac{p(-i\lambda_e)}{q(-i\lambda_e)} \right) \\ &= \lim_{\epsilon \downarrow 0} \left(\frac{2}{\pi} \left(\int_0^{\lambda_e - \epsilon} ds \frac{e^{-|s|t}|s|}{2(|s| - \lambda_e)} \frac{p(-i|s|)}{q(-i|s|)} + \int_{\lambda_e + \epsilon}^{+\infty} ds \frac{e^{-|s|t}|s|}{2(|s| - \lambda_e)} \frac{p(-i|s|)}{q(-i|s|)} \right) \right. \\ &\quad \left. - \frac{2}{\pi} \left(\int_0^{\lambda_e - \epsilon} ds \frac{e^{-\lambda_e t} \lambda_e^2}{s^2 - \lambda_e^2} \frac{p(-i\lambda_e)}{q(-i\lambda_e)} + \int_{\lambda_e + \epsilon}^{+\infty} \frac{e^{-\lambda_e t} \lambda_e^2}{s^2 - \lambda_e^2} \frac{p(-i\lambda_e)}{q(-i\lambda_e)} ds \right) \right), \end{aligned}$$

and the integrals in the second braces is vanishing for every ϵ . So, one remains with

$$J_2(t) = \frac{2}{\pi} P.V. \int_0^\infty \frac{e^{-st} s}{2(s - \lambda_e)} \frac{p(-is)}{q(-is)} ds$$

Finally, the rational function in the above integral can be decomposed in a singular (not integrable) part and an integrable one, obtaining

$$J_2(t) = \frac{1}{\pi} \mathcal{L} \left(\frac{c_1 s + c_2}{(s - \gamma_e)^2 + \omega_e^2} \right) + \frac{c_3}{\pi} P.V. \int_0^\infty \frac{e^{-st}}{s - \lambda_e} ds$$

where c_1 , c_2 and c_3 are constants depending on the parameters. \square

The first integral has the time behaviour of the other already studied contributions, and the last one is related to the exponential integral function:

$$\frac{c_3}{\pi} P.V. \int_0^\infty \frac{e^{-st}}{s - \lambda_e} ds = -\frac{c_3}{\pi} e^{-\lambda_e t} \text{Ei}(\lambda_e t)$$

From the well known asymptotic behaviour for large arguments of the exponential integral function, one deduce that the leading contribution for large times to the survival amplitude is of the order $\frac{1}{t}$ when $t \gg 1$. In summary, collecting the previous calculations, we end with the following result

Theorem 3.3. *There exist complex constants c_1, c_2, c_3 depending on the physical parameters $\mathbf{e}, m, c, \omega_0$ such that the survival amplitude is given by*

$$\mathcal{S}(t) = c_1 e^{-\lambda_e |t|} + c_2 e^{-\gamma_e |t|} e^{-i\omega_e t} + c_3 e^{-\lambda_e |t|} \text{Ei}(\lambda_e |t|).$$

3.3. Transition amplitudes. Now we turn to the evaluation of the transition amplitudes. As we saw in the previous theorem, the unperturbed excited states of the oscillator decay. The transition to a lower energy state take place with the emission of electromagnetic radiation, photons in the Fock space. To give a quantitative estimate of the probability of emission of photons one has to decide which functions describe the one particle states of the electromagnetic field one is interested in. We choose to calculate the transition amplitudes in the case of a regularized plane wave of (approximatively) given momentum.

Classically, a natural choice for the functions φ representing the photons of energy $\hbar\nu$ should be the divergence-free plane waves of the kind

$$\varphi(x) := k \wedge \zeta e^{-i\nu k \cdot x/c}, \quad |k| = 1.$$

Of course such functions are not square integrable and so we consider the divergence-free regularization defined by

$$\varphi^\epsilon(x) := \left(k - \frac{i\epsilon}{\nu} \frac{x}{|x|} \right) \wedge \zeta e^{-i\alpha k \cdot x/c - \epsilon|x|/c}, \quad \nu > 0, \epsilon > 0, |k| = 1.$$

We choose this one as the 1-particle wavefunctions describing photons. Eventually, we are interested in removing the cutoff ϵ from the amplitude transitions.

We anticipate the definitions of some quantities appearing in the statement and proof of the following result. To evaluate the resolvent between states relevant to the transitions, one need the scalar product $\langle \varphi^\epsilon, \mathcal{G}_z^\pm \rangle$. Assuming that k is along the z -axis and ζ is along the y -axis,

by elementary calculations one obtains

$$\begin{aligned}
\chi^\epsilon(\pm z) &:= \langle \varphi^\epsilon, \mathcal{G}_z^\pm \rangle \\
&\equiv k \wedge \zeta^* \frac{1}{\alpha} \left(\frac{\alpha^2 - \epsilon(\epsilon \mp iz)}{\alpha^2 + (\epsilon \mp iz)^2} - \frac{i\epsilon}{2\alpha^2} \ln \frac{\alpha + i\epsilon \pm z}{\alpha - i\epsilon \mp z} \right) \\
&\equiv (k \wedge \zeta^*)(\chi_r^\epsilon(\pm z) + \chi_l^\epsilon(\pm z)).
\end{aligned}$$

We have written the function $\chi^\epsilon(\pm z)$ just defined, as the sum $\chi_r^\epsilon(\pm z) + \chi_l^\epsilon(\pm z)$ with two summands, a rational one and a logarithmic one. The first, $\chi_r^\epsilon(\pm z)$ has poles at the points $\pm\alpha \pm i\epsilon$, and at the same points the logarithmic part $\chi_l^\epsilon(\pm z)$ has branching points. Due to this fact, some cautions are needed in the use of the residue theorem to evaluate transition amplitudes. Note, moreover, that these branching points of χ are to be thought as artificial byproducts of the regularization of the plane wave, and that the poles of the function χ_r correspond to the frequencies of the plane wave. These poles depend on the particular wavefunction representing the photon, at variance with the poles of the function $1/q(z)$, which are determined by the physical parameters of the system. We indicate with \mathcal{C} the logarithmic cut in the complex plane, and we distinguish the components of the cut with real part of a fixed sign (\pm) , with \mathcal{C}^\pm . With these premises, we state the following

Lemma 3.4. *For any $\zeta_1, \zeta_2 \in \mathbb{C}^3$ one has*

$$\langle (\varphi_1^\epsilon, 0), e^{-it(L_e^+)^{1/2}}(0, \zeta_2) \rangle = \mathcal{A}^\epsilon(t) k \cdot (\zeta_1^* \wedge \zeta_2),$$

where

$$\mathcal{A}^\epsilon(t) = -\frac{4\pi\mathbf{e}}{c} \frac{\kappa_2}{\kappa^2} I^\epsilon(t)$$

and, with $\pm t > 0$,

$$\begin{aligned}
I^\epsilon(t) = & \mp e^{\mp \lambda_e t} \lambda_e^2 \frac{\chi(-i\lambda_e)}{q(-i\lambda_e)} \pm e^{\mp \gamma_e t} \frac{e^{-i\omega_e t} 2z_\pm}{z_\pm + i\lambda_e} \frac{2z_\pm + i\lambda_e}{q'(z_\pm)} \chi(z_\pm) \\
& \pm e^{\mp \epsilon t} \frac{e^{\mp i\nu t} 2(\pm\nu - i\epsilon)}{\pm\nu + i(\lambda_e - \epsilon)} \frac{2\nu + i(\lambda_e - 2\epsilon)}{q(\pm\nu - i\epsilon)} \frac{\mp\nu - i\epsilon}{2\nu} \\
& + \frac{i}{\pi} \int_{\mathbb{R}} ds \left(\frac{e^{\mp \lambda_e t} \lambda_e^3}{s^2 - \lambda_e^2} \frac{\chi(-i\lambda_e)}{q(-i\lambda_e)} + \frac{e^{\mp |s|t} |s|(-2|s| + \lambda_e)}{2(|s| - \lambda_e)} \frac{\chi(-i|s|)}{q(-i|s|)} \right) \\
& + \frac{i}{\pi} \int_{\mathbb{R}} ds \frac{e^{\mp |s|t} |s|(2|s| + \lambda_e)}{2(|s| + \lambda_e)} \frac{\chi(i|s|)}{q(i|s|)} \\
& + \frac{\epsilon}{\nu^3} \int_C dz e^{-itz} \frac{2z + i\lambda_e}{z + i\lambda_e} \frac{1}{q(z)} \\
& + \pi \frac{\epsilon}{\nu^3} \left(\int_{C^+} dz e^{-itz} \frac{2z + i\lambda_e}{z + i\lambda_e} \frac{z}{q(z)} - \int_{C^-} dz e^{-itz} \frac{2z + i\lambda_e}{z + i\lambda_e} \frac{z}{q(z)} \right),
\end{aligned}$$

Proof. One has, proceeding as in the proof of Lemma 3.1

$$\begin{aligned}
& \langle (\varphi_1^\epsilon, 0), (L_e - z^2)^{-1}(0, \zeta_2)^+ \rangle \\
& = \frac{4\pi c \mathbf{e} \kappa_2}{\kappa^2(\mp iz + \lambda_e)} \frac{(\pm 2iz - \lambda_e)(\pm iz + \lambda_e)}{\left(\frac{2\mathbf{e}^2}{3c^3} z^2 - \frac{m\omega_0^2}{\lambda_e^2} (\mp iz + \lambda_e) \right) (\pm iz + \lambda_e)} \chi^\epsilon(\pm z) (k \wedge \zeta_1^*) \cdot \zeta_2.
\end{aligned}$$

Therefore

$$\langle (\varphi_1^\epsilon, 0), e^{-it(L_e^+)^{1/2}}(0, \zeta_2) \rangle = -\frac{4\pi \mathbf{e}}{c} \frac{\kappa_2}{\kappa^2} \left(I_1(t) + \frac{1}{\pi} \int_{\mathbb{R}} ds I_2(t, s) \right) (k \wedge \zeta_1^*) \cdot \zeta_2,$$

where (we omit the dependence on ϵ)

$$\begin{aligned}
I_1(t) &:= \frac{1}{2\pi i} \lim_{a \uparrow \infty} \int_{-a}^a d\lambda e^{-it\lambda} \lambda (g(\lambda) - g(-\lambda)), \\
I_2(t, s) &:= \frac{1}{2\pi i} \int_{\mathbb{R}} d\lambda e^{-it\lambda} \lambda^2 \frac{g(\lambda) - g(-\lambda)}{s^2 + \lambda^2}, \\
g(\lambda) &:= \frac{2\lambda + i\lambda_e}{\lambda + i\lambda_e} \frac{\chi(\lambda)}{q(\lambda)}.
\end{aligned}$$

Now, let us choose a path in the complex lower halfplane which has the real axis as the upper side, avoids the cuts of the function χ_l (say, the

straight half lines parametrized as $z = \pm\nu + u - i\epsilon$, $\pm u > 0$), and close itself at ∞ along a great circle, as in the previous lemma. There will be contributions due to the residues of the function $1/q(z)$, the residues of the function $\chi_r(z)$, and to the discontinuity of the logarithmic part of $\chi_l(z)$ along the cut. In the end, one obtains, when $\pm t > 0$,

$$\begin{aligned} I_1(t) = & \mp e^{\mp\lambda_{\mathbf{e}}t} \lambda_{\mathbf{e}}^2 \frac{\chi(-i\lambda_{\mathbf{e}})}{q(-i\lambda_{\mathbf{e}})} \pm e^{\mp\gamma_{\mathbf{e}}t} \sum_{k=1,2} \frac{e^{\mp(-1)^k i\omega_{\mathbf{e}}t} z_k}{z_k + i\lambda_{\mathbf{e}}} \frac{2z_k + i\lambda_{\mathbf{e}}}{q'(z_k)} \chi(z_k) \\ & \pm e^{\mp\epsilon t} \sum_{k=1,2} \frac{e^{\mp i\nu t} w_k}{w_k + i\lambda_{\mathbf{e}}} \frac{2w_k + i\lambda_{\mathbf{e}}}{q(w_k)} r_k + \int_{\mathcal{C}} \frac{2z + i\lambda_{\mathbf{e}}}{z + i\lambda_{\mathbf{e}}} \frac{e^{-itz}}{q(z)} dz, \end{aligned}$$

where z_{\pm} are the poles of q , $w_k := (-1)^k \nu - i\epsilon$ are the two poles of $\chi_r(z)$ and r_k the residues of χ_r , and finally \mathcal{C} is the path along the cut of the logarithmic term.

Moreover we have

$$\begin{aligned} I_2(t, s) = & i \frac{e^{\mp\lambda_{\mathbf{e}}t} \lambda_{\mathbf{e}}^3}{s^2 - \lambda_{\mathbf{e}}^2} \frac{\chi(-i\lambda_{\mathbf{e}})}{q(-i\lambda_{\mathbf{e}})} \\ & + e^{\mp\gamma_{\mathbf{e}}t} \sum_{k=1,2} \frac{z_k}{s^2 + z_k^2} \frac{e^{\mp(-1)^k i\omega_{\mathbf{e}}t} z_k}{z_k + i\lambda_{\mathbf{e}}} \frac{2z_k + i\lambda_{\mathbf{e}}}{q'(z_k)} \chi(z_k) \\ & + e^{\mp\epsilon t} \sum_{k=1,2} \frac{w_k}{s^2 + w_k^2} \frac{e^{\mp i\nu t} w_k}{w_k + i\lambda_{\mathbf{e}}} \frac{2w_k + i\lambda_{\mathbf{e}}}{q(w_k)} r_k \\ & + i e^{\mp|s|t} \frac{|s|}{2} \left(\frac{\mp 2|s| + \lambda_{\mathbf{e}}}{|s| \mp \lambda_{\mathbf{e}}} \frac{\chi(\mp i|s|)}{q(\mp i|s|)} + \frac{\pm 2|s| + \lambda_{\mathbf{e}}}{|s| \pm \lambda_{\mathbf{e}}} \frac{\chi(\pm i|s|)}{q(\pm i|s|)} \right) \\ & + \int_{\mathcal{C}} e^{-itz} \frac{2z + i\lambda_{\mathbf{e}}}{z + i\lambda_{\mathbf{e}}} \frac{z^2}{q(z)} \frac{1}{s^2 + z^2} dz. \end{aligned}$$

Therefore one obtains

$$\begin{aligned}
& I_1(t) + \frac{1}{\pi} \int_{\mathbb{R}} ds I_2(t, s) \\
&= \mp e^{\mp \lambda_e t} \lambda_e^2 \frac{\chi(-i\lambda_e)}{q(-i\lambda_e)} \pm e^{\mp \gamma_e t} \frac{e^{-i\omega_e t} 2z_k}{z_k + i\lambda_e} \frac{2z_k + i\lambda_e}{q'(z_k)} \chi(z_k) \\
&\pm e^{\mp \epsilon t} \frac{e^{\mp i\omega_e t} 2w_k}{w_k + i\lambda_e} \frac{2w_k + i\lambda_e}{q(w_k)} r_k \\
&+ \frac{i}{\pi} \int_{\mathbb{R}} ds \left(\frac{e^{\mp \lambda_e t} \lambda_e^3}{s^2 - \lambda_e^2} \frac{\chi(-i\lambda_e)}{q(-i\lambda_e)} + \frac{e^{\mp |s|t} |s| (-2|s| + \lambda_e)}{2(|s| - \lambda_e)} \frac{\chi(-i|s|)}{q(-i|s|)} \right) \\
&+ \frac{i}{\pi} \int_{\mathbb{R}} ds \frac{e^{\mp |s|t} |s| (2|s| + \lambda_e)}{2(|s| + \lambda_e)} \frac{\chi(i|s|)}{q(i|s|)} \\
&+ \frac{\epsilon}{\nu^3} \int_{\mathcal{C}} dz e^{-itz} \frac{2z + i\lambda_e}{z + i\lambda_e} \frac{1}{q(z)} \\
&+ \pi \frac{\epsilon}{\nu^3} \left(\int_{\mathcal{C}^+} dz e^{-itz} \frac{2z + i\lambda_e}{z + i\lambda_e} \frac{z}{q(z)} - \int_{\mathcal{C}^-} dz e^{-itz} \frac{2z + i\lambda_e}{z + i\lambda_e} \frac{z}{q(z)} \right)
\end{aligned}$$

where $k = 2$ if $t > 0$ and $k = 1$ if $t < 0$ and \mathcal{C}^{\pm} are the components of \mathcal{C} with $\pm \text{Sign}(\text{Re}(z)) > 0$ respectively.

□

Now we can give the time behaviour of the various terms, as in the previous theorem. A first group of terms is composed by the resonant contributions. One distinguish natural resonances, depending on the structural parameters only (the physical constants \mathbf{e} , m , c), and the resonance due to the incident photon. For $\epsilon \downarrow 0$ this last contribution reduces to a strictly oscillating term, as expected. A second group of terms is given by the s integrals. We give their behaviour for vanishing ϵ only. The calculation is similar to the one already given for the survival amplitude, and one has in the end Laplace transforms of rational

functions with poles at the resonances, producing other exponentials of the type already seen, and an exponential integral function. So the leading behaviour of the type t^{-1} survives to the $\epsilon \downarrow 0$ limit.

The last group of terms are the contour complex integrals. These could be analyzed asymptotically as Fourier integrals of rational functions, but they vanish as $\epsilon \downarrow 0$. Summarizing we can state the following

Theorem 3.5. *There exist complex constants C_1, C_2, C_3 depending on the physical parameters $m, \mathbf{e}, c, \omega_0, \nu$, and a function $R(t)$ such that*

$$\lim_{\epsilon \downarrow 0} \mathcal{A}^\epsilon(t) = C_1 e^{-\lambda_\epsilon |t|} + C_2 e^{-\gamma_\epsilon |t|} e^{-i\omega_\epsilon t} + C_3 e^{-i\nu t} + R(t)$$

with

$$R(t) = O(1/t), \quad |t| \gg 1.$$

APPENDIX A. QUANTIZATION OF ABSTRACT WAVE EQUATIONS

In this appendix we give a brief and self contained introduction to the quantization of second order abstract wave equations, along the lines traced by I.E.Segal in the fifties and sixties and with an emphasis on the aspects of direct concern with our work. We refer for details and different approaches to [27] and [15] as regards abstract wave equations and to [25], [5] as regards quantization.

Let $B : D(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be a injective self-adjoint operator on the real Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. We denote by \mathcal{H}^1 , the Hilbert space given by the domain of B with the scalar product $\langle \cdot, \cdot \rangle_1$ leading to the graph norm, i.e.

$$\langle \phi_1, \phi_2 \rangle_1 := \langle B\phi_1, B\phi_2 \rangle + \langle \phi_1, \phi_2 \rangle.$$

We then define the Hilbert space $\bar{\mathcal{H}}^1$ by completing the pre-Hilbert space $D(B)$ endowed with the the scalar product

$$[\phi_1, \phi_2]_1 := \langle B\phi_1, B\phi_2 \rangle .$$

We extend the self-adjoint operator B to $\bar{\mathcal{H}}^1$ by considering $\bar{B} : \bar{\mathcal{H}}^1 \rightarrow \mathcal{H}$, the closed bounded extension of the densely defined linear operator

$$B : \mathcal{H}^1 \subseteq \bar{\mathcal{H}}^1 \rightarrow \mathcal{H} .$$

Since B is self-adjoint one has $\text{Ran}(B)^\perp = \text{Ker}(B)$, so that, being B injective, $\text{Ran}(B)$ is dense in \mathcal{H} . Therefore we can define $\bar{B}^{-1} : \mathcal{H} \rightarrow \bar{\mathcal{H}}^1$ as the closed bounded extension of the densely defined linear operator

$$B^{-1} : \text{Ran}(B) \subseteq \mathcal{H} \rightarrow \bar{\mathcal{H}}^1 .$$

One can then verify that \bar{B} is boundedly invertible with inverse given by \bar{B}^{-1} .

Given \bar{B} we can now introduce the space $\bar{\mathcal{H}}^2$ defined by

$$\bar{\mathcal{H}}^2 := \{ \phi \in \bar{\mathcal{H}}_1 : \bar{B}\phi \in \mathcal{H}^1 \} .$$

On the Hilbert space $\bar{\mathcal{H}}^1 \oplus \mathcal{H}$ with scalar product

$$\langle \langle (\phi_1, \dot{\phi}_1), (\phi_2, \dot{\phi}_2) \rangle \rangle := \langle \bar{B}\phi_1, \bar{B}\phi_2 \rangle + \langle \dot{\phi}_1, \dot{\phi}_2 \rangle .$$

we define the linear operator

$$W_B : \bar{\mathcal{H}}_2 \oplus \mathcal{H}^1 \subseteq \bar{\mathcal{H}}^2 \oplus \mathcal{H} \rightarrow \bar{\mathcal{H}}^1 \oplus \mathcal{H}, \quad W_B(\phi, \dot{\phi}) := (\dot{\phi}, -B\bar{B}\phi) .$$

Theorem A.1. *The linear operator W_B is skew-adjoint.*

Being W_B skew-adjoint, by the Stone theorem it generates the strongly continuous one-parameter group of isometric operators e^{tW_B} . By defining $(\phi(t), \dot{\phi}(t)) := e^{tW_B}(\phi, \dot{\phi})$, one has that, in the case $\phi \in \bar{\mathcal{H}}^2$, $\dot{\phi} \in \mathcal{H}^1$, $\phi(t)$ is the unique strong solution of the Cauchy problem

$$\ddot{\phi}(t) = -B\bar{B}\phi(t)$$

$$\phi(0) = \phi, \quad \dot{\phi}(0) = \dot{\phi}.$$

Given an arbitrary real Hilbert space $(\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})$ we will denote by $(\mathcal{K}_c, \langle \cdot, \cdot \rangle_c)$ its standard complexification, i.e. $\mathcal{K}_c = \mathcal{K} \oplus \mathcal{K}$, the multiplication by the complex unity being defined by $i\psi := J\psi$, $J(\varphi_1, \varphi_2) := (-\varphi_2, \varphi_1)$, and

$$\langle \psi_1, \psi_2 \rangle_c := \langle \psi_1, \psi_2 \rangle_{\mathcal{K} \oplus \mathcal{K}} - i \langle \psi_1, J\psi_2 \rangle_{\mathcal{K} \oplus \mathcal{K}}.$$

Given any linear operator $A : D(A) \subseteq \mathcal{K} \rightarrow \mathcal{K}$ on \mathcal{K} , we define $A_c : D(A_c) \subset \mathcal{K}_c \rightarrow \mathcal{K}_c$ by $D(A_c) := D(A) \times D(A)$, $A_c(\varphi_1, \varphi_2) := (A\varphi_1, A\varphi_2)$. Conversely, given any linear operator $L : D(L) \subset \mathcal{K}_c \rightarrow \mathcal{K}_c$, $L(\varphi_1, \varphi_2) \equiv (L_1(\varphi_1, \varphi_2), L_2(\varphi_1, \varphi_2))$, we define $L_r : D(L_r) \subseteq \mathcal{K} \rightarrow \mathcal{K}$ by $D(L_r) := \{\phi \in \mathcal{K} : (\phi, 0) \in D(L)\}$, $L_r := L_1(\varphi, 0)$.

By the above definitions B_c is self-adjoint and $(W_B)_c = W_{B_c}$, where

$$W_{B_c} : \bar{\mathcal{H}}_c^2 \oplus \mathcal{H}_c^1 \subseteq \bar{\mathcal{H}}_c^2 \oplus \mathcal{H}_c \rightarrow \bar{\mathcal{H}}_c^1 \oplus \mathcal{H}_c, \quad W_{B_c}(\psi, \dot{\psi}) := (\dot{\psi}, -B_c \bar{B}_c \psi).$$

By the Stone formula one has the following

Lemma A.2.

$$e^{tW_B} = \lim_{a \uparrow \infty} \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{-a}^a d\lambda \, e^{-it\lambda} ((iW_{B_c} - (\lambda + i\epsilon))^{-1} - (iW_{B_c} - (\lambda - i\epsilon))^{-1})_r.$$

The following lemma translates the wave flow in a Schrödinger like one.

Lemma A.3. *The map*

$$C_B : \bar{\mathcal{H}}^1 \oplus \mathcal{H} \rightarrow \mathcal{H}_c, \quad C_B(\phi, \dot{\phi}) := (\bar{B}\phi, \dot{\phi}).$$

is unitary once one makes $\bar{\mathcal{H}}^1 \oplus \mathcal{H}$ a complex Hilbert space by introducing the complex structure $J_B(\phi, \dot{\phi}) := (-\bar{B}^{-1}\dot{\phi}, \bar{B}\phi)$ and by defining $i(\phi, \dot{\phi}) := J_B(\phi, \dot{\phi})$ and the scalar product

$$\langle\langle(\phi_1, \dot{\phi}_1), (\phi_2, \dot{\phi}_2)\rangle\rangle_B := \langle\langle(\phi_1, \dot{\phi}_1), (\phi_2, \dot{\phi}_2)\rangle\rangle - i\langle\langle(\phi_1, \dot{\phi}_1), J_B(\phi_2, \dot{\phi}_2)\rangle\rangle.$$

Moreover

$$e^{tW_B} = C_B^* e^{-itB_c} C_B.$$

Once we have transformed the abstract wave equation $\ddot{\phi}(t) = -B\bar{B}\phi(t)$, defined on the real Hilbert space $\bar{\mathcal{H}}^1 \oplus \mathcal{H}$, into the Schrödinger equation $i\dot{\psi} = B_c\psi$, defined on the complex Hilbert space \mathcal{H}_c , we can then (second) quantize it in the standard way. Let us define \mathbf{K} to be the bosonic Fock space over \mathcal{H}_c (see [22], section II.4, for the definition). For any $\psi \in \mathcal{H}$ we define the self-adjoint operator on \mathbf{K} given by the Segal field $\mathbf{S}(\psi) := \frac{1}{\sqrt{2}}(\mathbf{C}(\psi) + \mathbf{C}^*(\psi))$, where \mathbf{C} and \mathbf{C}^* denote the usual creation and destruction operators (see [23], section X.7, for the definition). Given

the Segal field \mathbf{S} we can then define the Weyl system $\mathbf{W}(\psi) := e^{i\mathbf{S}\psi}$, so that

$$\mathbf{W}(\psi_1 + \psi_2) = e^{\frac{i}{2}\langle\psi_1, \psi_2\rangle_c} \mathbf{W}(\psi_1) \mathbf{W}(\psi_2).$$

The unitary strongly continuous one-parameter group of evolution on \mathbf{K} defined by $\mathbf{U}(t) := \Gamma(e^{-itB_c})$ satisfies the relations

$$\mathbf{W}(e^{-itB_c}\psi) = \mathbf{U}(t)\mathbf{W}(\psi)\mathbf{U}(t)^*$$

and we denote by $d\Gamma(B_c)$ the self-adjoint operator on \mathbf{K} with generates $\mathbf{U}(t)$ (we refer to [23], section II.4, and [9] for the definitions of Γ and $d\Gamma$). The quantum hamiltonian corresponding to the (second) quantization of the abstract wave equation $\ddot{\phi}(t) = -B\bar{B}\phi(t)$ is defined by $\mathbf{H} := \hbar d\Gamma(B_c)$. Suppose now that we start with a selfadjoint operator A on the real Hilbert space \mathcal{K} . Denoting by A^+ the positive part of A and by \mathcal{K}^+ the projection of \mathcal{K} onto the positive spectral subspace. Then we can apply the previous construction to $B := (A^+)^{1/2}$, considered as an injective self-adjoint operator on $\mathcal{H} := \mathcal{K}^+$. However, since

$$\langle\psi_1, e^{-it(A^+)_c^{1/2}} \psi_2\rangle = \langle\psi_1^+, e^{-it(A^+)_c^{1/2}} \psi_2^+\rangle,$$

where ψ^+ denotes the projection of ϕ on \mathcal{K}^+ , we will work with $d\Gamma((A^+)_c^{1/2})$ on the bosonic Fock space over \mathcal{K} .

REFERENCES

- [1] S.Albeverio, F.Gesztesy, R.Hoegh-Krohn, H.Holden: *Solvable Models in Quantum Mechanics*. New York: Springer-Verlag 1988

- [2] A.Arai: Rigorous theory of spectra and radiation for a model in quantum electrodynamics. *J.Math.Phys.*, **24** (1983), 1896-1910
- [3] V.Bach, J.Frölich, I.M.Sigal: Quantum electrodynamics of confined nonrelativistic particles. *Adv. Math.*, **137** (1998), 299-395
- [4] V.Bach, J.Frölich, I.M.Sigal, Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field. *Commun. Math. Phys.*, **207** (1999), 249-290
- [5] J.Baez, I.E.Segal, Z.Zhou: *Introduction to Algebraic and Constructive Quantum Field Theory*. Princeton: Princeton University Press, 1992
- [6] H.A.Bethe: The Electromagnetic Shift of Energy Levels. *Phys. Rev.*, **72** (1947), 339-341
- [7] M.S. Birman: Existence Conditions for Wave Operators. *Am. Math. Soc. Trans., Ser. 2*, **54** (1966), 91-117
- [8] G.Breit, E.P. Wigner: Capture of slow neutrons. *Phys.Rev.*, **49** (1936), 519-531
- [9] J.M.Cook: The mathematics of second quantization. *Trans. Amer. Math. Soc.*, **74** (1953), 222-245
- [10] P.A.M.Dirac: Classical Theory of radiating electrons. *Proc. Roy.Soc.A*, **167** (1938), 148-167
- [11] E.Fermi: Quantum Theory of Radiation. *Rev. Mod. Phys.* **4** (1932), 87-132
- [12] M.Griesemer, E.H.Lieb, M.Loss: Ground states in nonrelativistic quantum electrodynamics. *Invent. Math.*, **145** (2001), 557-59
- [13] L.A.Khalfin: On the theory of the decay of a quasi-stationary state, Sov. Phys. Dokl. **2**, (1957) 340-344
- [14] L.A.Khalfin: Contribution to the decay theory of a quasi-stationay state, Sov. Phys. JETP **6**, (1958) 1053-1063
- [15] T.Kato: Scattering Theory with Two Hilbert Spaces. *J. Funct. Anal.*, **1** (1967), 342-369

- [16] E.H.Lieb, M.Loss: A bound on binding energies and mass renormalization in models of nonrelativistic quantum electrodynamics. *J.Stat.Phys.*, **108** (2002), 1057-1069
- [17] H.S.Lorentz: *The theory of Electrons*, 2nd Ed., New York: Dover 1952
- [18] M.Mück: Construction of metastable states in Quantum Electrodynamics, *Rev.Math.Phys.*, **16** (2004), 1-28
- [19] D.Noja, A.Posilicano: The Wave Equation with One Point Interaction and the (Linearized) Classical Electrodynamics of a Point Particle. *Ann. Inst. Henri Poincaré*, **68** (1998), 351–377
- [20] D.Noja, A.Posilicano: On the Point Limit of the Pauli–Fierz Model. *Ann. Inst. Henri Poincaré*, **71** (1999), 425-457
- [21] W.Pauli, M.Fierz: Zur Theorie der Emission Langwelliger Lichtquanten, *Nuovo Cimento*, **15** (1938), 168-188
- [22] M. Reed, B. Simon: *Methods of Modern Mathematical Physics. Vol. I: Functional Analysis*. New York, San Francisco, London: Academic Press 1980
- [23] M. Reed, B. Simon: *Methods of Modern Mathematical Physics. Vol. II: Fourier Analysis, Self-Adjointness*. New York, San Francisco, London: Academic Press 1975
- [24] M. Reed, B. Simon: *Methods of Modern Mathematical Physics. Vol. IV: Theory of Operators*. New York, San Francisco, London: Academic Press 1978
- [25] I.E.Segal: *Mathematical Problems of Relativistic Physics*. Providence, Rhode Island: AMS 1963
- [26] H.Spohn: Asymptotic completeness for Rayleigh scattering *J.Math.Phys.*, **38** (1997), 2281-2296
- [27] B. Weiss: Abstract Vibrating Systems. *J. Math. and Mech.*, **17** (1967), 241-255

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